

Final - Probability II (2025-26)

Time: 3 hours.

Attempt all questions. The total marks is 52 but the maximum you can score is 50.

1. The joint probability density function of random variables X and Y is given by

$$f(x, y) = 24xy, \quad 0 < x < 1, 0 < y < 1, 0 < x + y < 1.$$

- (a) Are X and Y independent? Explain. [1 mark]
 - (b) Compute the density function of X . [3 marks]
 - (c) Find the conditional density of Y given $X = x$. [2 marks]
 - (d) Compute $E[Y|X = x]$. [3 marks]
 - (e) Compute $E[Y]$. [3 marks]
2. Let $X_n, n \geq 1$ and X be random variables on a probability space. Say that $X_n \xrightarrow{L^1} X$ if $\mathbb{E}[|X_n - X|] \rightarrow 0$ as $n \rightarrow \infty$.
- (a) Show that $X_n \xrightarrow{L^1} X$ implies $X_n \xrightarrow{P} X$. [4 marks]
 - (b) Give an example to show that $X_n \xrightarrow{P} X$ does not imply that $X_n \xrightarrow{L^1} X$. [5 marks]
3. Let X_1, X_2, \dots, X_n be a set of i.i.d. continuous random variables with CDF F and density f , and let $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ denote their ordered values.
- (a) Compute the CDFs of $X_{(n)}$ and $X_{(1)}$. [4 marks]
 - (b) Compute the densities of $X_{(n)}$ and $X_{(1)}$. [4 marks]
 - (c) If X is independent of all the X_i 's and also has CDF F , then compute $P(X_{(1)} < X < X_{(n)})$. [4 marks]
4. Let X and Y be independent and identically distributed $\text{Unif}(0, 1)$ random variables. Find the joint density of $U = X + Y, V = X/(X + Y)$. [5 marks]
5. (a) State the central limit theorem. [3 marks]
- (b) Let $X_1, X_2, \dots, X_{1000}$ be i.i.d. Bernoulli(0.3) random variables, that is $X_i = 1$ with probability 0.3 and $X_i = 0$ with probability 0.7. Use the central limit theorem to give an approximation of

$$P\left(2000 \leq \sum_{i=1}^{1000} X_i \leq 5000\right)$$

in terms of the CDF Φ of the standard normal distribution. [4 marks]

6. Let U_1, U_2, \dots be i.i.d. $\text{Unif}(0, 1)$ random variables and let

$$X_n = \left(\prod_{i=1}^n U_i\right)^{\frac{1}{n}}.$$

Show that $X_n \xrightarrow{P} e^{-1}$. [7 marks]